

Quantum Gravity with Boundaries near Two Dimensions

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Abstract

We evaluate the quantum corrections of the Einstein-Hilbert action with boundaries in the $2 + \epsilon$ dimensional expansion approach. We find the Einstein-Hilbert action with boundaries to be renormalizable to the one loop order. We compute the geometric entropy beyond the semiclassical approximation. It is found that the exact geometric entropy is related to the string susceptibility by the analytic continuation in the central charge. Our results also show that we can renormalize the divergent quantum corrections for the Bekenstein-Hawking entropy of blackholes by the gravitational coupling constant renormalization beyond two dimensions.

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1 Introduction

In quantum gravity, we need to study the influence of boundaries in many physically interesting questions. We may site the event horizons in blackhole physics and space-like hyper surfaces in quantum cosmology. Such questions also arise when we study the loop amplitudes in two dimensional gravity and open string theory.

In the blackhole spacetime, the existence of the event horizon leads to the very interesting physics such as the Bekenstein-Hawking entropy and the Hawking radiation. In the Euclidean blackhole spacetime, the event horizon is mapped to a point and the spacetime inside the event horizon simply does not exist. The periodicity of the Euclidean spacetime (rotation angle around the event horizon) implies that the system is thermal.

From the Minkowski point of view, we need to integrate out the physical degrees of freedom inside the event horizon. Such an integration leads to a mixed state. In ref. [1], it is shown that the blackhole entropy is given semiclassically by the Einstein-Hilbert action associated with the infinitesimal disc around the event horizon.

When we compute the quantum corrections to Bekenstein-Hawking entropy, it diverges since the divergence of the Einstein-Hilbert action form arises in the effective action. The difficulty of quantum gravity is the nonrenormalizability of the theory beyond two dimensions. However it can be renormalized by the $2 + \epsilon$ dimensional expansion approach. Furthermore the theory possesses the short distance fixed point with proper matter contents and consistent quantum gravity theory may be constructed. Therefore the study of the renormalization of the geometric entropy in the $2 + \epsilon$ dimensional quantum gravity must be illuminating.

With these physical motivations, we study the renormalization of the Einstein-Hilbert action with boundaries in the $2 + \epsilon$ dimensions.

2 1-Loop Renormalization

We shall evaluate the quantum corrections of the Einstein-Hilbert action with boundaries in the $2 + \epsilon$ dimensional quantum gravity. Here we adopt the background field method, which gives a gauge invariant effective action. It is shown in this section that the divergences are also of the Einstein-Hilbert action form. In the first sub-section, we compute the bulk

contributions to the effective action and explain our computational method. In the following sub-section, we compute the boundary contributions.

2.1 Contributions from The Bulk

We first calculate the quantum corrections of the Einstein-Hilbert action when a $2 + \epsilon$ dimensional manifold M is not bounded. They are the bulk contributions proportional to the Einstein action. As it is expected, we reproduce the well-known result of the conformal anomaly of two dimensional quantum gravity in the $2 + \epsilon$ dimensional expansion approach.

Let us consider the action of a free scalar field in a curved space:

$$- \int d^D x \sqrt{\hat{g}} \frac{1}{2} \varphi \hat{\Delta} \varphi, \quad (1)$$

where $\hat{\Delta}$ is the Laplacian in the curved space. It is defined in terms of the metric of the curved background $\hat{g}_{\mu\nu}$ as

$$\hat{\Delta} \equiv \frac{1}{\sqrt{\hat{g}}} \frac{\partial}{\partial x^\mu} \sqrt{\hat{g}} \hat{g}^{\mu\nu} \frac{\partial}{\partial x^\nu}. \quad (2)$$

Here x^μ is a set of local coordinates.

Since we would like to obtain the 1-loop local divergences, we only need to consider the short-distance propagation of a particle. It depends not on the global property of the manifold but on the local one. So we can adopt the local coordinate method. When the particle propagates for a very short time, it feels as if it were moving on the almost flat Euclidean space. Therefore we can perturb the theory around the flat-space one.

In this paper, we adopt the Riemann's normal coordinates. The advantage of the method is that we can consider the local property of the manifold in a manifestly covariant way. In such coordinates, the Laplacian in the curved background $\hat{\Delta}$ is expanded covariantly as follows:

$$\begin{aligned} \hat{\Delta} &= \left(\frac{\partial}{\partial u^\mu} \right)^2 + \frac{1}{3} \hat{R}^\mu{}_\rho{}^\nu{}_\sigma u^\rho u^\sigma \frac{\partial^2}{\partial u^\mu \partial u^\nu} + \frac{2}{3} \hat{R}_\mu{}^\nu u^\mu \frac{\partial}{\partial u^\nu} + O(\hat{R}^2), \\ &\equiv \Delta + P(u). \end{aligned} \quad (3)$$

Here u^μ is a set of the geodesic coordinates from a given point on the manifold M . The Riemann and Ricci curvatures are evaluated at the origin of the normal coordinates $u^\mu = 0$. Δ and $P(u)$ denote the flat space Laplacian and the perturbation around it, respectively.

In general, we can obtain the 1-loop effective action by integrating the quadratic terms of the action.

$$\begin{aligned}\Gamma_{\text{matter}} &= \frac{1}{2} \log \text{Det} \left(\frac{\Delta + P}{\Delta} \right), \\ &= \frac{1}{2} \text{Tr} [\log \{ -(\Delta + P) \} - \log(-\Delta)] ,\end{aligned}\quad (4)$$

We can reexpress the above by introducing a proper time τ as follows [2]:

$$\Gamma_{\text{matter}} = -\frac{1}{2} \int_{-\infty}^{\infty} d^D x \sqrt{\hat{g}} \int_0^{\infty} \frac{d\tau}{\tau} [\langle x | e^{-\tau\{-(\Delta+P)\}} | x \rangle - \langle x | e^{-\tau(-\Delta)} | x \rangle]. \quad (5)$$

Here, $\hat{G}(x_1, x_2; \tau) \equiv \langle x_1 | e^{-\tau\{-(\Delta+P)\}} | x_2 \rangle$ is called a heat kernel. This is because it is the Green's function of the heat equation

$$\left\{ \frac{\partial}{\partial \tau} - (\Delta_{x_1} + P(x_1)) \right\} \hat{G}(x_1, x_2; \tau) = \delta(\tau) \delta^{(D)}(x_1 - x_2) / \sqrt{\hat{g}(x_2)}. \quad (6)$$

On the other hand, $G(x_1, x_2; \tau) \equiv \langle x_1 | e^{-\tau(-\Delta)} | x_2 \rangle$ is the flat space Green's function, satisfying

$$\left(\frac{\partial}{\partial \tau} - \Delta_{x_1} \right) G(x_1, x_2; \tau) = \delta(\tau) \delta^{(D)}(x_1 - x_2) / \sqrt{\hat{g}(x_2)}. \quad (7)$$

Its solution is easily obtained by Fourier transformations.

$$\begin{aligned}G(x_1, x_2; \tau) &= \frac{1}{\sqrt{\hat{g}(x_2)}} \int_{-\infty}^{\infty} \frac{d^D p}{(2\pi)^D} e^{-\tau p^2} e^{ip \cdot (x_1 - x_2)} \\ &= \frac{1}{\sqrt{\hat{g}(x_2)}} \frac{1}{(4\pi\tau)^{\frac{D}{2}}} e^{-\frac{(x_1 - x_2)^2}{4\tau}}\end{aligned}\quad (8)$$

We can obtain \hat{G} perturbatively, in terms of the flat space Green's function G .

$$\hat{G} = G + GPG + \dots \quad (9)$$

As a result, we only have to evaluate the following integration.

$$\begin{aligned}\Gamma_{\text{matter}} &= -\frac{1}{2} \int_{-\infty}^{\infty} d^D x \sqrt{\hat{g}} \int_0^{\infty} \frac{d\tau}{\tau} \\ &\quad \times \left[\int_0^{\infty} d\tau_1 d\tau_2 \delta(\tau - \tau_1 - \tau_2) \int_{-\infty}^{\infty} d^D x' G(x, x'; \tau_1) P(x') G(x', x; \tau_2) + \dots \right].\end{aligned}\quad (10)$$

The calculation of the above is straightforward, where it is convenient to choose x as the origin of the normal coordinate expansion. We find the result of the 1-loop divergence of a real scalar field in $D = 2 + \epsilon$ dimensions as

$$\Gamma_{\text{matter}} \simeq -\frac{1}{24\pi\epsilon} \int_M \hat{R} \sqrt{\hat{g}} d^D x. \quad (11)$$

Next we shall consider the gravitational and ghost fields.

We adopt the parametrization of the gravitational degrees of freedom by Kawai, Kitazawa and Ninomiya [3], which singles out the conformal mode of the metric ϕ :

$$\begin{aligned} g_{\mu\nu} &\equiv \tilde{g}_{\mu\nu} e^{-\phi}, \\ &\equiv \hat{g}_{\mu\rho} (e^h)^\rho{}_\nu e^{-\phi}, \end{aligned} \quad (12)$$

where $\hat{g}_{\mu\nu}$ is the background metric and $h_{\mu\nu}$ is a traceless symmetric matrix ($\hat{g}^{\mu\nu} h_{\mu\nu} = 0$). The tensor indices are raised or lowered by the background metric. In such a parametrization, the Einstein action near two dimensions becomes

$$\begin{aligned} \frac{\mu^\epsilon}{G} \int d^D x \sqrt{g} R &= \frac{\mu^\epsilon}{G} \int d^D x \sqrt{\hat{g}} \hat{R} \\ &+ \frac{\mu^\epsilon}{G} \int d^D x \sqrt{\hat{g}} \left\{ \frac{1}{4} \hat{\nabla}_\rho h^\mu{}_\nu \hat{\nabla}^\rho h^\nu{}_\mu + \frac{1}{2} \hat{R}^\sigma{}_{\mu\nu\rho} h^\rho{}_\sigma h^{\mu\nu} \right. \\ &- \frac{\epsilon}{4} (D-1) \hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{\epsilon^2}{8} \phi^2 \hat{R} + \frac{\epsilon}{2} \phi h^\mu{}_\nu \hat{R}^\nu{}_\mu \\ &\left. - \frac{1}{2} \hat{\nabla}_\mu h^\mu{}_\rho \hat{\nabla}_\nu h^{\nu\rho} + \frac{\epsilon}{2} \phi \hat{\nabla}_\mu \hat{\nabla}_\nu h^{\mu\nu} \right\} + \dots, \end{aligned} \quad (13)$$

where G , μ are the gravitational coupling constant and a renormalization scale to define it, respectively.

In order to cancel the last two terms, we adopt a Feynman-like gauge:

$$\frac{\mu^\epsilon}{G} \int d^D x \sqrt{\hat{g}} \frac{1}{2} (\hat{\nabla}_\mu h^\mu{}_\rho + \frac{\epsilon}{2} \partial_\rho \phi) (\hat{\nabla}_\nu h^{\nu\rho} + \frac{\epsilon}{2} \partial^\rho \phi). \quad (14)$$

The change of the metric under the general coordinate transformation is

$$\delta g_{\mu\nu} = \partial_\mu \epsilon^\rho g_{\rho\nu} + g_{\mu\rho} \partial_\nu \epsilon^\rho + \epsilon^\rho \partial_\rho g_{\mu\nu}. \quad (15)$$

It leads the gauge transformations of $h^\mu{}_\nu$ and ϕ fields as:

$$\begin{aligned} \delta h^\mu{}_\nu &= \hat{\nabla}^\mu \epsilon_\nu + \hat{\nabla}_\nu \epsilon^\mu - \frac{2}{D} \hat{\nabla}_\rho \epsilon^\rho \delta^\mu{}_\nu + \dots, \\ \delta \phi &= \epsilon^\mu \partial_\mu \phi - \frac{2}{D} \hat{\nabla}_\mu \epsilon^\mu + \dots. \end{aligned} \quad (16)$$

Following the standard procedure, we find the ghost action to be

$$\frac{\mu^\epsilon}{G} \int d^D x \sqrt{\hat{g}} (\bar{\eta}_\mu \hat{\nabla}_\nu \hat{\nabla}^\nu \eta^\mu - \hat{R}^\mu{}_\nu \bar{\eta}_\mu \eta^\nu - \frac{\epsilon}{2} \partial_\nu \phi \hat{\nabla}^\mu \bar{\eta}_\mu \eta^\nu + \dots). \quad (17)$$

In this way, we find the quadratic terms needed for the 1-loop calculations in the background gauge.

$$\begin{aligned}
& \frac{\mu^\epsilon}{G} \int d^D x \sqrt{\hat{g}} \left[\frac{1}{4} (\delta^\mu{}_\rho \delta^\nu{}_\sigma - \frac{1}{D} \hat{g}^{\mu\nu} \hat{g}_{\rho\sigma}) \hat{\nabla}_\alpha h_{\mu\nu} \hat{\nabla}^\alpha h^{\rho\sigma} + \frac{1}{2} \hat{R}^\sigma{}_{\mu\nu\rho} h^\rho{}_\sigma h^{\mu\nu} \right. \\
& \quad \left. - \frac{\epsilon}{8} D \hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{\epsilon^2}{8} \hat{R} \phi^2 \right. \\
& \quad \left. - \delta^\mu{}_\nu \hat{\nabla}_\alpha \bar{\eta}_\mu \hat{\nabla}^\alpha \eta^\nu - \hat{R}^\mu{}_\nu \bar{\eta}_\mu \eta^\nu \right].
\end{aligned} \tag{18}$$

As a result, we can evaluate the Green's functions for $h_{\mu\nu}$, ϕ and ghost fields.

$$\begin{aligned}
& [\hat{I}^{\mu\nu}{}_{,\alpha\beta}(x_1) \frac{\partial}{\partial \tau} - \{ \hat{I}^{\mu\nu}{}_{,\alpha\beta}(x_1) \hat{\Delta}_{x_1} + 2 I^{\mu\nu}{}_{,\gamma\delta}(x_1) \hat{R}^\gamma{}_\alpha{}^\delta{}_\beta(x_1) \}] \hat{G}^{\alpha\beta}{}_{,\rho\sigma}(x_1, x_2; \tau) \\
& \quad = \hat{I}^{\mu\nu}{}_{,\rho\sigma}(x_2) \delta(\tau) \delta^{(D)}(x_1 - x_2) / \sqrt{\hat{g}(x_2)}, \\
& [\frac{\partial}{\partial \tau} - (\hat{\Delta}_{x_1} + \frac{\epsilon}{D} \hat{R}(x_1))] \hat{G}_\phi(x_1, x_2; \tau) = \delta(\tau) \delta^{(D)}(x_1 - x_2) / \sqrt{\hat{g}(x_2)}, \\
& [\delta^\mu{}_\rho \frac{\partial}{\partial \tau} - (\delta^\mu{}_\rho \hat{\Delta}_{x_1} - \hat{R}^\mu{}_\rho(x_1))] \hat{G}^\rho{}_\nu(x_1, x_2; \tau) = \delta^\mu{}_\nu \delta(\tau) \delta^{(D)}(x_1 - x_2) / \sqrt{\hat{g}(x_2)},
\end{aligned} \tag{19}$$

where $\hat{I}^{\mu\nu}{}_{,\alpha\beta}(x) = \frac{1}{2} \delta^\mu{}_\alpha \delta^\nu{}_\beta + \frac{1}{2} \delta^\mu{}_\beta \delta^\nu{}_\alpha - \frac{1}{D} \hat{g}^{\mu\nu}(x) \hat{g}_{\alpha\beta}(x)$ is the identity for the traceless symmetric tensors in a D dimensional curved space. We have normalized the heat kernels so that the coefficients of the Laplacians are equal to one. It is allowed to do so since we consider the normalization independent ratio as in the eqn. (4).

We note that the boundary terms do not appear in the eqs. (19), since we assume that $h_{\mu\nu}$, ϕ and ghost fields fall off rapidly enough at $x \rightarrow \infty$. In the next sub-section, we consider the case where the manifold M has a boundary. It will be seen that we also obtain the eqs. (19). This is because the boundary term which arises when we take the variation of the Einstein term cancels out that of the extrinsic curvature term.

In a similar way to a scalar field case, we can obtain the heat kernel for $h_{\mu\nu}$, ϕ and ghost fields in a curved background as $\hat{G}^{\mu\nu}{}_{,\rho\sigma}$, \hat{G}_ϕ and $\hat{G}^\mu{}_\nu$ respectively. Here, it is convenient to choose x_2 as the origin of the normal coordinates $u^\mu = 0$ and to assign the normal coordinates u^μ to x_1 . We note that the geometrical quantities evaluated at u are expressed in terms of those evaluated at the origin $u^\mu = 0$, as follows:

$$\begin{aligned}
\hat{g}_{\mu\nu}(u) &= \delta_{\mu\nu} - \frac{1}{3} \hat{R}_{\mu\rho\nu\sigma}(0) u^\rho u^\sigma + \dots, \\
\hat{R}_{\mu\nu\rho\sigma}(u) &= \hat{R}_{\mu\nu\rho\sigma}(0) + \dots, \\
\hat{R}_{\mu\nu}(u) &= \hat{R}_{\mu\nu}(0) + \dots, \\
\hat{R}(u) &= \hat{R}(0) + \dots.
\end{aligned} \tag{20}$$

Here, the dots express the higher order terms, which are unnecessary for us to calculate divergent contributions for the effective action. It is important to note that the Riemann and Ricci curvatures at u are equal to those at the origin up to this order. In the following, we simply express $\hat{R}_{\mu\nu\rho\sigma}(0), \dots$ as $\hat{R}_{\mu\nu\rho\sigma}, \dots$. In terms of them, we get the following 1-loop divergences from ϕ , $h_{\mu\nu}$ and ghost fields.

$$\begin{aligned}\Gamma_\phi &\simeq -\frac{1}{24\pi\epsilon} \int_M \hat{R} \sqrt{\hat{g}} d^D x, \\ \Gamma_h &\simeq \left(-\frac{2}{24\pi\epsilon} + \frac{1}{2\pi\epsilon}\right) \int_M \hat{R} \sqrt{\hat{g}} d^D x, \\ \Gamma_{\text{ghost}} &\simeq \left(-\frac{-4}{24\pi\epsilon} + \frac{1}{2\pi\epsilon}\right) \int_M \hat{R} \sqrt{\hat{g}} d^D x.\end{aligned}\tag{21}$$

The conformal mode gives the identical contribution with that of a scalar field. It is due to the fact that, for the conformal mode, the perturbation proportional to \hat{R} is $O(\epsilon)$ smaller than the kinetic term.

Consequently, we obtain the total 1-loop divergences of the theory from the bulk:

$$\Gamma_{\text{div.}} = \frac{25-c}{24\pi\epsilon} \int_M \hat{R} \sqrt{\hat{g}} d^D x.\tag{22}$$

We need to add the counter term to cancel this divergence. However the counter term breaks the conformal invariance of the otherwise conformally invariant theory. This is the origin of the well known conformal anomaly of two dimensional quantum gravity in our approach.

2.2 Contributions from The Boundary

In this sub-section, we consider a D -dimensional manifold M bounded by a $(D-1)$ -dimensional smooth boundary ∂M . The corrections to the 1-loop divergence (22) due to the existence of the boundary is proportional to the extrinsic curvature of the manifold. The combination of the bulk and boundary contributions turns out to be of the Einstein-Hilbert action form.

In the vicinity of the boundary, it is convenient to specialize the coordinates of an interior point P by a new coordinate set (w, x^i) [$i = 1, \dots, D-1$]. The first coordinate w is the geodesic distance from P to ω , which is the projection of P on the boundary, and the other

$D - 1$ coordinates x^i characterize the position of ω on the boundary. We further specialize the coordinates x^i of ω , using a set y^i of Riemann's normal coordinates from a given point ω_0 on the boundary.

In this set of coordinates, the metric has only the diagonal components, and the Laplacian (2) is expanded as:

$$\hat{\Delta} = \frac{\partial^2}{\partial w^2} + \left(\frac{\partial}{\partial y^i}\right)^2 - \frac{D-1}{R} \frac{\partial}{\partial w} + 2w \sum_{i=1}^{D-1} \frac{1}{R_i} \left(\frac{\partial}{\partial y^i}\right)^2 + \dots, \quad (23)$$

where R_i are the main curvature radii of the boundary at ω_0 , and R is the mean curvature defined by

$$\frac{1}{R} \equiv \frac{1}{D-1} \sum_{i=1}^{D-1} \frac{1}{R_i}. \quad (24)$$

As it will be seen in the following, the corrections of the unperturbed Green's function $G(x_1, x_2; \tau)$ due to the existence of a boundary are exponentially damped when x_1 and x_2 move away from the boundary. So we only have to consider the vicinity of the boundary to evaluate the influence of it. Since we would like to obtain the local divergences, we need not consider the long-distance propagation of a particle. If the particle propagates near the boundary for a very short time, it behaves as if the boundary were flat. Therefore we can well approximate the unperturbed Green's function by that of a half Euclidean space:

$$\begin{aligned} G(x_1, x_2; \tau) &= G_0(w_1, y_1; w_2, y_2; \tau) \mp G_0(w_1, y_1; -w_2, y_2; \tau), \\ &\equiv G_0(x_1, x_2; \tau) + G_1(x_1, x_2; \tau), \end{aligned} \quad (25)$$

where G_0 is the free space Green's function, and the signs $-$ and $+$ correspond to the Dirichlet and Neumann boundary conditions respectively. This is because the signs $-$ and $+$ make $G(x_1, x_2; \tau)$ to be anti-symmetric and symmetric respectively under the inversion of the signs of the coordinate w . G_1 is the correction of the unperturbed Green's function due to the existence of the boundary. The explicit forms of G_0 and G_1 are given by

$$\begin{aligned} G_0(w_1, y_1; w_2, y_2; \tau) &= \frac{1}{\sqrt{\hat{g}(x_2)}} \int \frac{dq d^{D-1}p}{(2\pi)^D} e^{-\tau(q^2+p^2)} e^{iq(w_1-w_2)} e^{ip(y_1-y_2)}, \\ &= \frac{1}{\sqrt{\hat{g}(x_2)}} \frac{1}{(4\pi\tau)^{\frac{D}{2}}} e^{-\frac{(w_1-w_2)^2}{4\tau}} e^{-\frac{(y_1-y_2)^2}{4\tau}}, \\ G_1(w_1, y_1; w_2, y_2; \tau) &= \mp \frac{1}{\sqrt{\hat{g}(x_2)}} \int \frac{dq d^{D-1}p}{(2\pi)^D} e^{-\tau(q^2+p^2)} e^{iq(w_1+w_2)} e^{ip(y_1-y_2)}, \\ &= \mp \frac{1}{\sqrt{\hat{g}(x_2)}} \frac{1}{(4\pi\tau)^{\frac{D}{2}}} e^{-\frac{(w_1+w_2)^2}{4\tau}} e^{-\frac{(y_1-y_2)^2}{4\tau}}. \end{aligned} \quad (26)$$

We note that the free space Green's function G_0 has the translational invariance in the flat space limit, while the correction G_1 does not in the direction perpendicular to the boundary. The G_1 decreases exponentially as the distance from the boundary increases. Since only the $w = 0$ cotributes in the large momentum limit($q, p \rightarrow \infty$) or the short time limit($\tau \rightarrow 0$), we obtain the divergences due to the presence of the boundary from G_1 .

In terms of this unperturbed Green's function, we can extract the corrections of Green's function due to the existence of the boundary by subtracting the perturbative expansions of the free space Green's function from those of the Green's function of the bounded space [4].

$$\begin{aligned} \delta\hat{G}(x_1, x_2; \tau) &= G_1(x_1, x_2; \tau) + \int_0^\infty d\tau_1 d\tau_2 \delta(\tau - \tau_1 - \tau_2) \int_0^\infty dw' \int_{-\infty}^\infty d^{D-1}y' \sqrt{\hat{g}(x')} \\ &\times [-G_0(x_1, \bar{x}'; \tau_1) P(\bar{x}') G_0(\bar{x}', x_2; \tau_2) + G_0(x_1, x'; \tau_1) P(x') G_1(x', x_2; \tau_2) \\ &+ G_1(x_1, x'; \tau_1) P(x') G_0(x', x_2; \tau_2) + G_1(x_1, x'; \tau_1) P(x') G_1(x', x_2; \tau_2)] + \dots, \end{aligned} \quad (27)$$

where $x = (w, y^i)$, $\bar{x} = (-w, y^i)$ and $P(x')$ is the perturbation given by the 3rd and 4th terms of the r.h.s. of (23) .

We can now evaluate the 1-loop divergences from the boundary due to a free scalar field, using $\delta\hat{G}$,

$$\begin{aligned} \delta\Gamma_{\text{matter}} &= -\frac{1}{2} \int_0^\infty dw \int_{-\infty}^\infty d^{D-1}y \sqrt{\hat{\gamma}} \int_0^\infty \frac{d\tau}{\tau} [\delta\hat{G}(w, y; w, y; \tau) - G_1(w, y; w, y; \tau)], \\ &\simeq \frac{1}{12\pi\epsilon} \int_{\partial M} \hat{K} \sqrt{\hat{\gamma}} d^{D-1}x. \end{aligned} \quad (28)$$

\hat{K} and $\hat{\gamma}$ are the extrinsic curvature of the boundary and the restriction of the metric to the boundary, respectively. \hat{K} is defined in terms of the inward unit normal vector n^i as $\hat{K} = \hat{\gamma}_j^i \hat{\nabla}_i n^j$. We have used the relation between the mean curvature and the extrinsic curvature: $(D-1)/R = \hat{K}$. In the both Dirichlet and Neumann boundary conditions, we obtain the above result. Therefore the sum of the bulk and boundary contributions due to a free scalar field results in

$$\Gamma_{\text{matter}} \simeq -\frac{1}{24\pi\epsilon} \left(\int_M \hat{R} \sqrt{\hat{g}} d^D x - 2 \int_{\partial M} \hat{K} \sqrt{\hat{\gamma}} d^{D-1}x \right). \quad (29)$$

This combination of the scalar curvature \hat{R} and the extrinsic curvature \hat{K} reminds us of the Gauss-Bonnet theorem:

$$\chi(M) = -\frac{1}{4\pi} \left(\int_M \hat{R} \sqrt{\hat{g}} d^2 x - 2 \int_{\partial M} \hat{K} \sqrt{\hat{\gamma}} dx \right), \quad (30)$$

which gives a topological invariant of two dimensional manifolds: the Euler number. Indeed, the classical action for the matter is conformally invariant in two dimensions. This is the

reason why we have obtained the divergence which becomes the topological invariant in the two dimensional limit.

Next we shall evaluate the corrections from gravitational and ghost fields. When a manifold is bounded by a $(D - 1)$ -dimensional sub-manifold, we have to add a surface term to the action with the Dirichlet boundary condition to obtain the Einstein's field equation as the classical equation of the action [5].

$$I = \frac{\mu^\epsilon}{G} [\int_M R \sqrt{\hat{g}} d^D x - 2 \int_{\partial M} K \sqrt{\hat{\gamma}} d^{D-1} x + (\text{gauge fixing and ghost terms})]. \quad (31)$$

Here the linear terms of $h_{\mu\nu}$ fields and the conformal mode are dropped since the background fields satisfy Einstein's field equation, which is obtained by considering the variation of the action with the Dirichlet boundary conditions for $h_{\mu\nu}$ fields and the conformal mode. We note that the heat equations for $h_{\mu\nu}$ fields and the conformal mode are the same as those in the unbounded manifold (19) respectively, due to the surface term and the Dirichlet boundary condition.

Using the Green's function (27), we can calculate the boundary contribution from $h_{\mu\nu}$ fields:

$$\delta\Gamma_h \simeq \left(\frac{2}{24\pi\epsilon} - \frac{1}{2\pi\epsilon} \right) \cdot 2 \int_{\partial M} \hat{K} \sqrt{\hat{\gamma}} d^{D-1} x. \quad (32)$$

The second part of the above expression comes from the tad pole divergence at the boundary. We have used the fact that the Gauss-Bonnet combination is free from the boundary contribution. The divergences from the conformal mode is identical to that from a free scalar field as in the bulk contribution.

For ghost fields, we should also choose the Dirichlet boundary condition. To see this, it is convenient to adopt the normal coordinates explained in the above. In those coordinates, we can easily see that $\partial_\mu h^{\mu\nu} = 0$ on ∂M since $h_{\mu\nu}$ fields are diagonalized as $h_{00} = 0$, $h_{ij} = -\frac{2w}{R_i} \delta_{ij} + \dots$ ($i, j = 1, \dots, D - 1$). So we obtain $\bar{\eta}_\mu = 0$ on ∂M from the following relation between the gauge fixing and ghost terms.

$$\delta_B(\bar{\eta}_\nu \partial_\mu h^{\mu\nu}) = \frac{1}{2} (\partial_\mu h^{\mu\nu})^2 - \bar{\eta}_\nu \partial_\mu (\delta_B h^{\mu\nu}), \quad (33)$$

where δ_B denotes the BRS transformation. Choosing the Dirichlet boundary condition, we also obtain the same heat equation for ghost fields as (19).

In a similar fashion to the $h_{\mu\nu}$ field's case, we can calculate the boundary contribution from the ghost field.

$$\delta\Gamma_{\text{ghost}} = \left(\frac{-4}{24\pi\epsilon} - \frac{1}{2\pi\epsilon}\right) \cdot 2 \int_{\partial M} \hat{K} \sqrt{\hat{\gamma}} d^{D-1}x. \quad (34)$$

We note that the sums of the bulk and boundary divergences of ϕ , $h_{\mu\nu}$ and ghost fields take the Euler class forms in the two dimensional limit, respectively.

$$\begin{aligned} \Gamma_\phi &\simeq -\frac{1}{24\pi\epsilon} \left(\int_M \hat{R} \sqrt{\hat{g}} d^Dx - 2 \int_{\partial M} \hat{K} \sqrt{\hat{\gamma}} d^{D-1}x \right), \\ \Gamma_h &\simeq \left(-\frac{2}{24\pi\epsilon} + \frac{1}{2\pi\epsilon} \right) \left(\int_M \hat{R} \sqrt{\hat{g}} d^Dx - 2 \int_{\partial M} \hat{K} \sqrt{\hat{\gamma}} d^{D-1}x \right), \\ \Gamma_{\text{ghost}} &\simeq \left(-\frac{-4}{24\pi\epsilon} + \frac{1}{2\pi\epsilon} \right) \left(\int_M \hat{R} \sqrt{\hat{g}} d^Dx - 2 \int_{\partial M} \hat{K} \sqrt{\hat{\gamma}} d^{D-1}x \right). \end{aligned} \quad (35)$$

Consequently we obtain the total 1-loop divergence from c copies of scalar fields, $h_{\mu\nu}$, ϕ and ghost fields:

$$\Gamma_{\text{div.}} = \frac{25-c}{24\pi\epsilon} \left(\int_M \hat{R} \sqrt{\hat{g}} d^Dx - 2 \int_{\partial M} \hat{K} \sqrt{\hat{\gamma}} d^{D-1}x \right). \quad (36)$$

These divergent terms are the extension of the result in the unbounded case (22). They have the form proportional to the Einstein-Hilbert action with boundaries. We note that they are also proportional to the Euler class in the two dimensional limit. It is naturally expected since only the BRS trivial parts of the action (31) break the conformal invariance in two dimensions.

The bare action with the counter term is

$$I_0 = \frac{1}{G_0} \left[\int_M R \sqrt{g} d^Dx - 2 \int_{\partial M} K \sqrt{\gamma} d^{D-1}x \right]. \quad (37)$$

where the bare gravitational coupling is $\frac{1}{G_0} = \mu^\epsilon \left(\frac{1}{G} - \frac{25-c}{24\pi\epsilon} \right)$. Therefore we can compensate the divergence by renormalizing the gravitational coupling constant and need not introduce an additional parameter to the theory.

3 Conclusions

We have evaluated the quantum corrections of the Einstein-Hilbert action with boundaries in the $2 + \epsilon$ dimensional expansion approach. The $2 + \epsilon$ dimensional manifold M is assumed to have a $1 + \epsilon$ dimensional smooth boundary.

We have imposed the Dirichlet or Neumann boundary conditions for the matter fields. When we consider the quantum fluctuations around the classical background $\hat{g}_{\mu\nu}$, we are led to choose the Dirichlet boundary conditions for $h_{\mu\nu}$ field and the conformal mode. This is because the equation of motion for the classical background becomes the Einstein's field equation only when we choose the Dirichlet boundary conditions for $h_{\mu\nu}$ field and the conformal mode. It is found from the BRS formalism that we should also impose the Dirichlet boundary condition for ghost fields.

We have studied the 1-loop corrections of the Einstein-Hilbert action with boundaries from the matter, $h_{\mu\nu}$, ϕ and ghost fields. The divergences are also of the Einstein-Hilbert action form with boundaries. Therefore the divergences are removed by the renormalization of the gravitational coupling constant.

Our result has an application to the Bekenstein-Hawking entropy of blackholes. As mentioned in the introduction, the entropy of blackholes is given by the Einstein-Hilbert action associated with the infinitesimal discs. The $2 + \epsilon$ dimensional expansion approach shows that one loop divergence of the Einstein-Hilbert action form arises. Therefore the quantum corrections for the blackhole entropy are also divergent. However it is also clear that we can obtain the finite quantum corrections for the blackhole entropy by renormalizing the gravitational coupling constant.

The Euclidean blackhole spacetime in D dimensions has the topology $R^2 \times S^\epsilon$ where S^ϵ is a ϵ dimensional sphere. The blackhole entropy is the Euler class of a small disk centered at the horizon multiplied by the area A_ϵ of the S^ϵ there[1]:

$$S_{BH} = \frac{4\pi}{G_0} A_\epsilon, \quad (38)$$

which becomes the standard formula if we adopt the standard convention $G_0 \rightarrow 16\pi G_0$. The renormalization group improved semiclassical entropy of the blackhole is

$$S_{BH} = \frac{4\pi\mu^\epsilon}{G(\mu)} A_\epsilon, \quad (39)$$

where we have replaced the Newton constant by the running coupling constant. It is natural to choose the renormalization scale μ to match the blackhole scale such that $\mu^\epsilon A_\epsilon = 1$. Then the renormalized blackhole entropy changes with the scale of the Blackhole as

$$\mu \frac{d}{d\mu} S_{BH} = -(\epsilon - \frac{25-c}{24\pi} G) S_{BH}, \quad (40)$$

where we have used the renormalization group equation for $1/G$ [3].

In the literature, the entropy of the blackholes and closely related geometric entropy have been studied[6, 7, 8]. Our results are certainly consistent with these results. In particular the conformal mode dependence of the geometric entropy is studied in [8]. In our approach, the conformal mode dependence of the geometric entropy comes from the counter term. It is the only source of the conformal mode dependence in two dimensions since the tree action is conformally invariant in two dimensional limit.

Let us consider the geometric entropy of a manifold with a closed boundary. The variation of the entropy with respect to the scale transformation is:

$$\begin{aligned}\delta S &= -\delta I_0 \\ &= \delta\phi\left(\frac{\epsilon}{2G} - \frac{25-c}{48\pi}\right)\mu^\epsilon\left[\int_M R \sqrt{\hat{g}} d^D x - 2 \int_{\partial M} K \sqrt{\hat{\gamma}} d^{D-1}x\right]\end{aligned}\quad (41)$$

This formula is consistent with the renormalization group equation (40). By taking the two dimensional limit, we obtain

$$\frac{\delta S}{\delta\phi} = \frac{25-c}{12}.\quad (42)$$

The constant mode of ϕ is related to the area of a disc as

$$\int_M \exp(-\alpha\phi) d^2x = A,\quad (43)$$

where we also have to renormalize the cosmological constant operator in fully quantum theory[9]. Hence we find $\phi \sim -\frac{1}{\alpha}\log A$. The requirement of the conformal invariance determines $\alpha = \frac{25-c}{12} - \frac{\sqrt{(1-c)(25-c)}}{12}$ for $c < 1$. For $c > 25$, $\alpha = \frac{25-c}{12} + \frac{\sqrt{(c-1)(c-25)}}{12}$. These formulas possess the correct semiclassical limit for large $|c|$. They are related by the analytic continuation in c . We find the scale dependence of the exact geometric entropy as

$$\frac{\delta S}{\delta\log A} = -\alpha \frac{25-c}{12}.\quad (44)$$

This result agrees with [8] in the leading order of c . Here again we have a difficulty to interpret the theory for $1 < c < 25$.

Comparing to the semiclassical results, the physical meaning of (44) is much more transparent for $c < 1$. In two dimensions, the Gauss-Bonnet action is topological. Therefore the induced Liouville action represents the entropy of the theory. Our results has followed from the same quantum effect. It can be interpreted as the quantum entropy in association with the two dimensional disc with a fixed area. In fact it is nothing but the string susceptibility for $c < 1$. The geometric entropy for $c > 25$ can be obtained by the analytic continuation in

c. The difficulty to quantize the theory with $c > 25$ in Euclidean spacetime is the conformal mode instability. On the other hand the entropy is defined in Euclidean spacetime. The conformal mode instability always exists beyond two dimensions in the semiclassical regime.

Therefore it is likely that the concept of the blackhole entropy and Hawking radiation are valid only in the semiclassical approximation. Although we have studied geometric entropy beyond the semiclassical approximation, we have to contemplate the physical implications of our investigations. Nevertheless we expect that the whole physical picture holds as a very good approximation in the weak coupling regime. Then it certainly makes sense to ask what is the temperature of such a quasithermal object as a blackhole. We expect that our results are valid in such a physical interpretation.

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